CS 188: Artificial Intelligence Spring 2009

Lecture 4: Constraint Satisfaction
1/29/2009

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Slides adapted from Dan Klein, Stuart Russell or Andrew Moore

Announcements

- The Python tutorial (Project 0) was due...
- Project 1 (Search) is due next Wednesday
  - Find partners at end of lecture
  - Food search is hard; traveling salesman and ants
- Written assignment 1 will be out tomorrow
  - Printed copies will be handed out in section
  - Due Tuesday, February 10th at the beginning of lecture (or section the day before)

Today

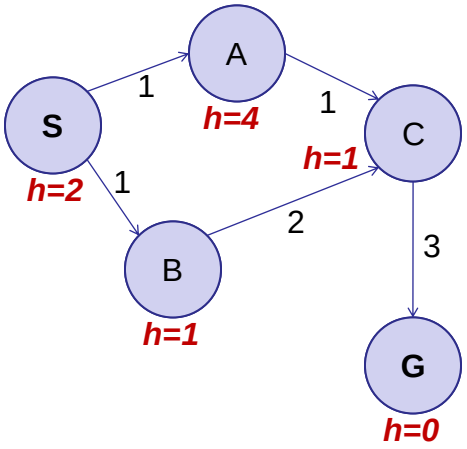
- A* Graph Search Recap
- Constraint Satisfaction Problems

A* Review

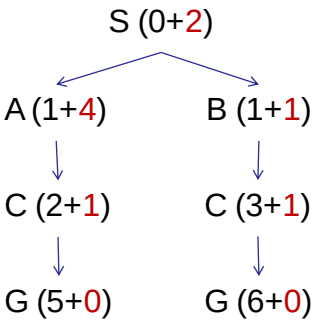
- A* uses both backward costs g and forward estimate h: f(n) = g(n) + h(n)
- A* tree search is optimal with admissible heuristics (optimistic future cost estimates)
- Heuristic design is key: relaxed problems can help

A* Graph Search Gone Wrong

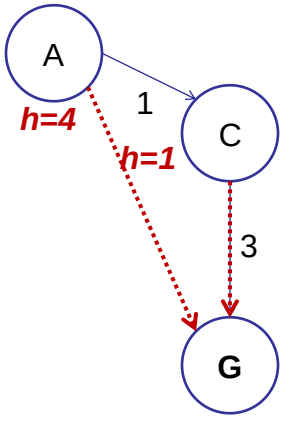
State space graph



Search tree



Consistency



The story on Consistency:

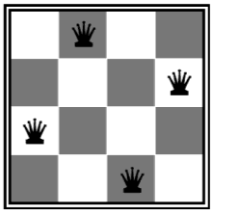
- Definition:
  cost(A to C) + h(C) ≥ h(A)
- Consequence in search tree:
  Two nodes along a path: N_A, N_C
  $g(N_C) = g(N_A) + \text{cost(A to C)}$
  $g(N_C) + h(C) \geq g(N_A) + h(A)$
- The f value along a path never decreases
- Non-decreasing f means you're optimal to every state (not just goals)

Optimality Summary

- Tree search:
  - A* optimal if heuristic is admissible (and non-negative)
  - Uniform Cost Search is a special case (h = 0)
- Graph search:
  - A* optimal if heuristic is consistent
  - UCS optimal (h = 0 is consistent)
- In general, natural admissible heuristics tend to be consistent
- Remember, costs are always positive in search!

Constraint Satisfaction Problems

- Standard search problems:
  - State is a "black box": arbitrary data structure
  - Goal test: any function over states
  - Successor function can be anything
- Constraint satisfaction problems (CSPs):
  - A special subset of search problems
  - State is defined by variables X_i with values from a domain D (sometimes D depends on i)
  - Goal test is a set of constraints specifying allowable combinations of values for subsets of variables
- Simple example of a *formal representation language*
- Allows useful general-purpose algorithms with more power than standard search algorithms



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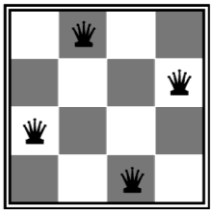
Example: N-Queens

- Formulation 1:
  - Variables: X_{ij}
  - Domains: $\{0, 1\}$
  - Constraints



$$\forall i, j, k \;\; (X_{ij}, X_{ik}) \in \{(0,0), (0,1), (1,0)\}$$
$$\forall i, j, k \;\; (X_{ij}, X_{kj}) \in \{(0,0), (0,1), (1,0)\}$$
$$\forall i, j, k \;\; (X_{ij}, X_{i+k,j+k}) \in \{(0,0), (0,1), (1,0)\}$$
$$\forall i, j, k \;\; (X_{ij}, X_{i+k,j-k}) \in \{(0,0), (0,1), (1,0)\}$$
$$\sum_{i,j} X_{ij} = N$$

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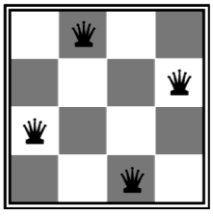
Example: N-Queens

- Formulation 2:
  - Variables: Q_k
  - Domains: $\{11, 12, 13, \ldots 21, \ldots NN\}$
  - Constraints:



$$\forall i, j \;\; \text{non-threatening}(Q_i, Q_j)$$
$$\forall i, j \;\; (Q_i, Q_j) \in \{(11, 23), (11, 24), \ldots\}$$

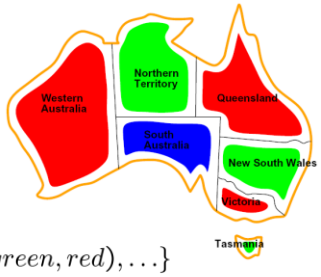
… there's an even better way! What is it?

Example: Map-Coloring

- Variables: $WA,\ NT,\ Q,\ NSW,\ V,\ SA,\ T$
- Domain: $D = \{red, green, blue\}$
- Constraints: adjacent regions must have different colors

$$WA \neq NT$$

$$(WA, NT) \in \{(red, green), (red, blue), (green, red), \ldots\}$$



- Solutions are assignments satisfying all constraints, e.g.:

$$\{WA = red, NT = green, Q = red,$$
$$NSW = green, V = red, SA = blue, T = green\}$$

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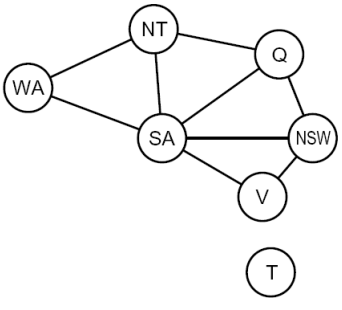
Constraint Graphs

- Binary CSP: each constraint relates (at most) two variables
- Constraint graph: nodes are variables, arcs show constraints
- General-purpose CSP algorithms use the graph structure to speed up search. E.g., Tasmania is an independent subproblem!



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Example: Cryptarithmetic

- Variables:

  $F\ T\ U\ W\ R\ O\ X_1\ X_2\ X_3$

- Domains:

  $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$

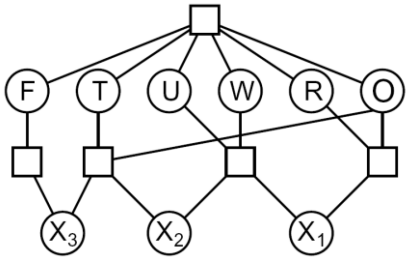
- Constraints:

  $\text{alldiff}(F, T, U, W, R, O)$

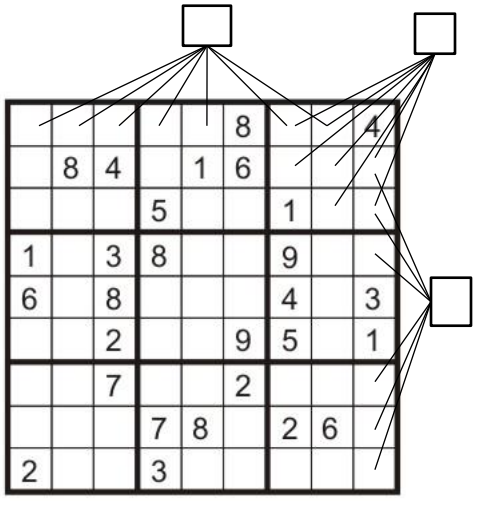
  $O + O = R + 10 \cdot X_1$

  $\cdots$

```
  T W O
+ T W O
-------
F O U R
```



Example: Sudoku



- Variables:
  - Each open square
- Domains:
  - {1,2,…,9)
- Constraints:

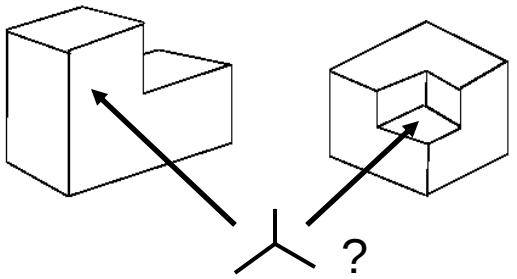
  9-way alldiff for each column

  9-way alldiff for each row

  9-way alldiff for each region

Example: The Waltz Algorithm

- The Waltz algorithm is for interpreting line drawings of solid polyhedra
- An early example of a computation posed as a CSP



- Look at all intersections
- Adjacent intersections impose constraints on each other

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Varieties of CSPs

- Discrete Variables
  - Finite domains
    - Size d means $O(d^n)$ complete assignments
    - E.g., Boolean CSPs, including Boolean satisfiability (NP-complete)
  - Infinite domains (integers, strings, etc.)
    - E.g., job scheduling, variables are start/end times for each job
    - Need a *constraint language*, e.g., $StartJob_1 + 5 < StartJob_3$
    - Linear constraints solvable, nonlinear undecidable
- Continuous variables
  - E.g., start/end times for Hubble Telescope observations
  - Linear constraints solvable in polynomial time by LP methods (see cs170 for a bit of this theory)

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Varieties of Constraints

- Varieties of Constraints
  - Unary constraints involve a single variable (equiv. to shrinking domains):

    $$SA \neq green$$

  - Binary constraints involve pairs of variables:

    $$SA \neq WA$$

  - Higher-order constraints involve 3 or more variables:
    e.g., cryptarithmetic column constraints

- Preferences (soft constraints):
  - E.g., red is better than green
  - Often representable by a cost for each variable assignment
  - Gives constrained optimization problems
  - (We'll ignore these until we get to Bayes' nets)

Real-World CSPs

- Assignment problems: e.g., who teaches what class
- Timetabling problems: e.g., which class is offered when and where?
- Hardware configuration
- Spreadsheets
- Transportation scheduling
- Factory scheduling
- Floorplanning

- Many real-world problems involve real-valued variables…

Standard Search Formulation

- Standard search formulation of CSPs (incremental)
- Let's start with the straightforward, dumb approach, then fix it
- States are defined by the values assigned so far
  - Initial state: the empty assignment, {}
  - Successor function: assign a value to an unassigned variable
  - Goal test: the current assignment is complete and satisfies all constraints
- Simplest CSP ever: two bits, constrained to be equal

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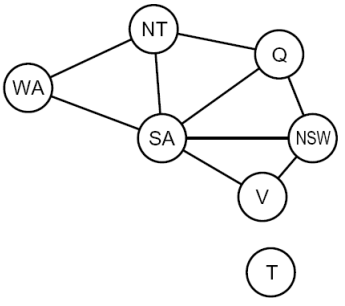
Search Methods

- What does BFS do?
- What does DFS do?
  - [demo]



- What's the obvious problem here?
- What's the slightly-less-obvious problem?

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Backtracking Search

- Idea 1: Only consider a single variable at each point:
  - Variable assignments are commutative
  - I.e., [WA = red then NT = green] same as [NT = green then WA = red]
  - Only need to consider assignments to a single variable at each step
  - How many leaves are there?
- Idea 2: Only allow legal assignments at each point
  - I.e. consider only values which do not conflict previous assignments
  - Might have to do some computation to figure out whether a value is ok
- Depth-first search for CSPs with these two improvements is called *backtracking search* (useless name, really)
  - [ANIMATION]
- Backtracking search is the basic uninformed algorithm for CSPs
- Can solve n-queens for n ≈ 25

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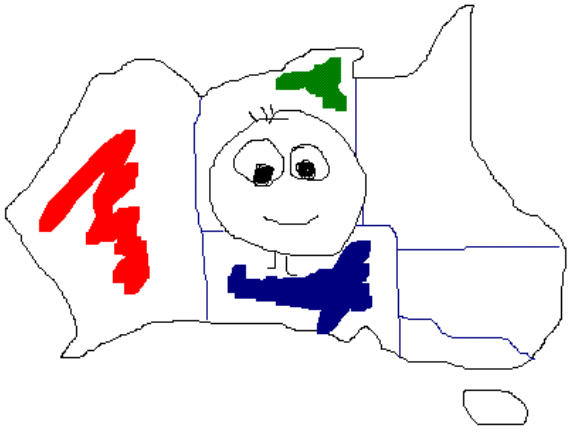
5 Minute Break



Courtesy of Dan Gillick

Backtracking Search

```
function BACKTRACKING-SEARCH(csp) returns solution/failure
  return RECURSIVE-BACKTRACKING({ }, csp)

function RECURSIVE-BACKTRACKING(assignment, csp) returns soln/failure
  if assignment is complete then return assignment
  var ← SELECT-UNASSIGNED-VARIABLE(VARIABLES[csp], assignment, csp)
  for each value in ORDER-DOMAIN-VALUES(var, assignment, csp) do
      if value is consistent with assignment given CONSTRAINTS[csp] then
          add {var = value} to assignment
          result ← RECURSIVE-BACKTRACKING(assignment, csp)
          if result ≠ failure then return result
          remove {var = value} from assignment
  return failure
```

- What are the choice points?

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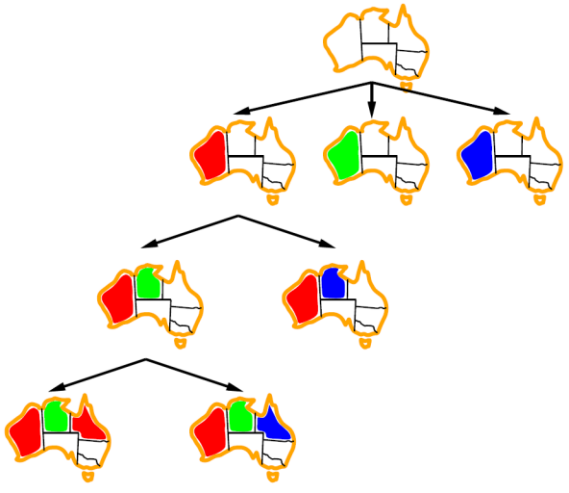
Backtracking Example



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Improving Backtracking

- General-purpose ideas can give huge gains in speed:
  - Which variable should be assigned next?
  - In what order should its values be tried?
  - Can we detect inevitable failure early?
  - Can we take advantage of problem structure?

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Minimum Remaining Values

- Minimum remaining values (MRV):
  - Choose the variable with the fewest legal values

- Why min rather than max?
- Called most constrained variable
- "Fail-fast" ordering

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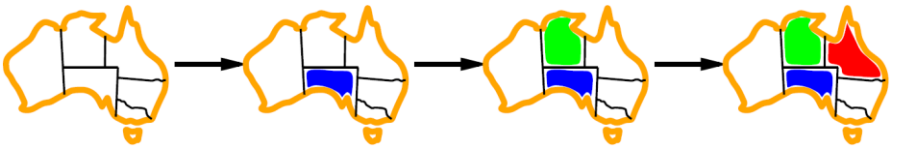
Degree Heuristic

- Tie-breaker among MRV variables
- Degree heuristic:
  - Choose the variable participating in the most constraints on remaining variables



- Why most rather than fewest constraints?

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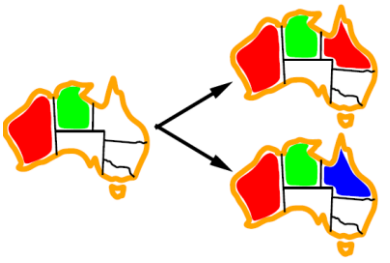
Least Constraining Value

- Given a choice of variable:
  - Choose the *least constraining value*
  - The one that rules out the fewest values in the remaining variables
  - Note that it may take some computation to determine this!
- Why least rather than most?
- Combining these heuristics makes 1000 queens feasible



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Forward Checking

- Idea: Keep track of remaining legal values for unassigned variables (using immediate constraints)
- Idea: Terminate when any variable has no legal values

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Constraint Propagation

- Forward checking propagates information from assigned to unassigned variables, but doesn't provide early detection for all failures:

- NT and SA cannot both be blue!
- Why didn't we detect this yet?
- *Constraint propagation* repeatedly enforces constraints (locally)

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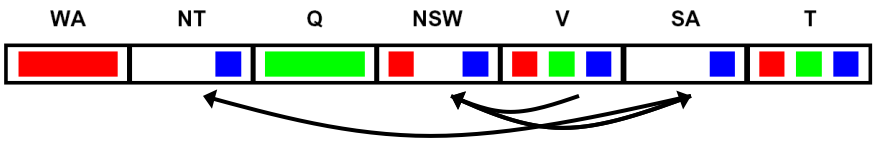
Arc Consistency

- Simplest form of propagation makes each arc *consistent*
  - X → Y is consistent iff for *every* value x there is *some* allowed y



- If X loses a value, neighbors of X need to be rechecked!
- Arc consistency detects failure earlier than forward checking
- What's the downside of arc consistency?
- Can be run as a preprocessor or after each assignment

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Arc Consistency

function AC-3(*csp*) **returns** the CSP, possibly with reduced domains
  inputs: *csp*, a binary CSP with variables $\{X_1, X_2, \ldots, X_n\}$
  local variables: *queue*, a queue of arcs, initially all the arcs in *csp*

  while *queue* is not empty **do**
    $(X_i, X_j) \leftarrow$ REMOVE-FIRST(*queue*)
    if REMOVE-INCONSISTENT-VALUES(X_i, X_j) **then**
      for each X_k **in** NEIGHBORS[X_i] **do**
        add (X_k, X_i) to *queue*

---

function REMOVE-INCONSISTENT-VALUES(X_i, X_j) **returns** true iff succeeds
  removed ← *false*
  for each x **in** DOMAIN[X_i] **do**
    if no value y in DOMAIN[X_j] allows (x,y) to satisfy the constraint $X_i \leftrightarrow X_j$
      then delete x from DOMAIN[X_i]; *removed* ← *true*
  return *removed*

- Runtime: $O(n^2d^3)$, can be reduced to $O(n^2d^2)$
- … but detecting all possible future problems is NP-hard – why?

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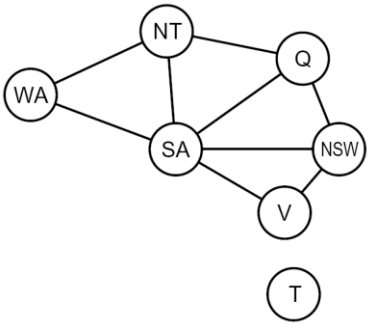
[demo forward checking and arc consistency]

Problem Structure

- Tasmania and mainland are independent subproblems
- Identifiable as connected components of constraint graph
- Suppose each subproblem has c variables out of n total
- Worst-case solution cost is $O((n/c)(d^c))$, linear in n
  - E.g., n = 80, d = 2, c =20
  - 2^{80} = 4 billion years at 10 million nodes/sec
  - $(4)(2^{20})$ = 0.4 seconds at 10 million nodes/sec



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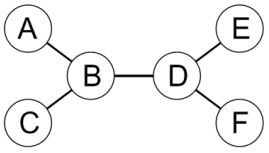
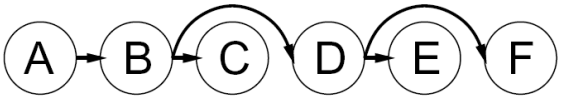
Tree-Structured CSPs

- Choose a variable as root, order variables from root to leaves such that every node's parent precedes it in the ordering





- For i = n : 2, apply RemoveInconsistent(Parent(X_i),X_i)
- For i = 1 : n, assign X_i consistently with Parent(X_i)
- Runtime: $O(n\ d^2)$

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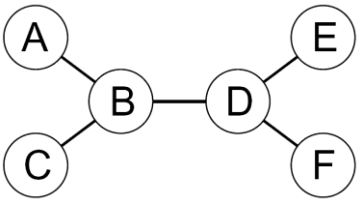
Tree-Structured CSPs



- Theorem: if the constraint graph has no loops, the CSP can be solved in $O(n\ d^2)$ time!
  - Compare to general CSPs, where worst-case time is $O(d^n)$
- This property also applies to logical and probabilistic reasoning: an important example of the relation between syntactic restrictions and the complexity of reasoning.

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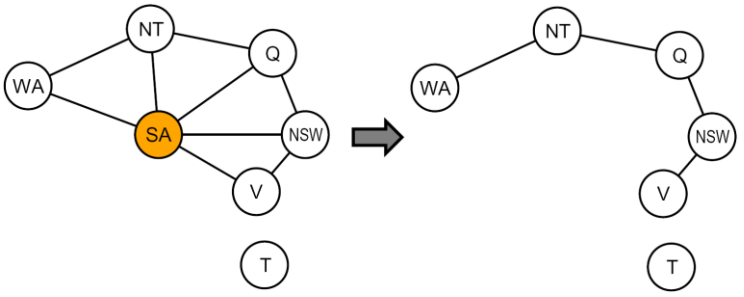
Nearly Tree-Structured CSPs



- Conditioning: instantiate a variable, prune its neighbors' domains
- Cutset conditioning: instantiate (in all ways) a set of variables such that the remaining constraint graph is a tree
- Cutset size c gives runtime $O((d^c) (n-c) d^2)$, very fast for small c

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Iterative Algorithms for CSPs

- Greedy and local methods typically work with "complete" states, i.e., all variables assigned
- To apply to CSPs:
  - Allow states with unsatisfied constraints
  - Operators *reassign* variable values
- Variable selection: randomly select any conflicted variable
- Value selection by min-conflicts heuristic:
  - Choose value that violates the fewest constraints
  - I.e., hill climb with h(n) = total number of violated constraints

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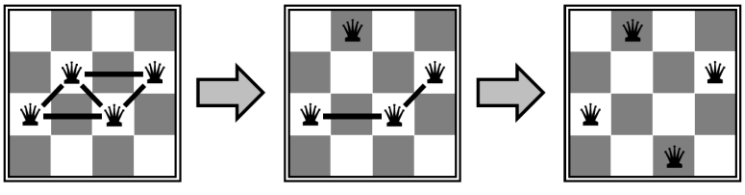
Example: 4-Queens



- States: 4 queens in 4 columns (4^4 = 256 states)
- Operators: move queen in column
- Goal test: no attacks
- Evaluation: h(n) = number of attacks

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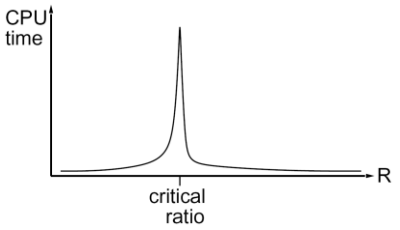
Performance of Min-Conflicts

- Given random initial state, can solve n-queens in almost constant time for arbitrary n with high probability (e.g., n = 10,000,000)
- The same appears to be true for any randomly-generated CSP *except* in a narrow range of the ratio

$$R = \frac{\text{number of constraints}}{\text{number of variables}}$$



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Summary

- CSPs are a special kind of search problem:
  - States defined by values of a fixed set of variables
  - Goal test defined by constraints on variable values
- Backtracking = depth-first search with one legal variable assigned per node
- Variable ordering and value selection heuristics help significantly
- Forward checking prevents assignments that guarantee later failure
- Constraint propagation (e.g., arc consistency) does additional work to constrain values and detect inconsistencies
- The constraint graph representation allows analysis of problem structure
- Tree-structured CSPs can be solved in linear time
- Iterative min-conflicts is usually effective in practice

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Local Search Methods

- Queue-based algorithms keep fallback options (backtracking)
- Local search: improve what you have until you can't make it better
- Generally much more efficient (but incomplete)

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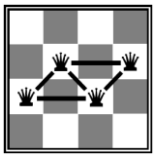
Types of Problems

- Planning problems:
  - We want a path to a solution (examples?)
  - Usually want an optimal path
  - *Incremental formulations*

- Identification problems:
  - We actually just want to know what the goal is (examples?)
  - Usually want an optimal goal
  - *Complete-state formulations*
  - Iterative improvement algorithms



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Hill Climbing

- Simple, general idea:
  - Start wherever
  - Always choose the best neighbor
  - If no neighbors have better scores than current, quit
- Why can this be a terrible idea?
  - Complete?
  - Optimal?
- What's good about it?

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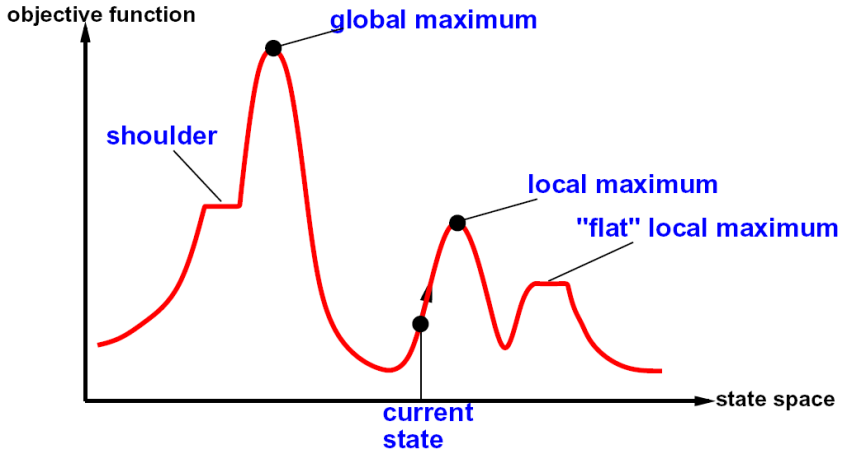
Hill Climbing Diagram



- Random restarts?
- Random sideways steps?

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Simulated Annealing

- Idea: Escape local maxima by allowing downhill moves
  - But make them rarer as time goes on

```
function SIMULATED-ANNEALING(problem, schedule) returns a solution state
  inputs: problem, a problem
          schedule, a mapping from time to "temperature"
  local variables: current, a node
                   next, a node
                   T, a "temperature" controlling prob. of downward steps

  current ← MAKE-NODE(INITIAL-STATE[problem])
  for t ← 1 to ∞ do
      T ← schedule[t]
      if T = 0 then return current
      next ← a randomly selected successor of current
      ΔE ← VALUE[next] – VALUE[current]
      if ΔE > 0 then current ← next
      else current ← next only with probability e^(Δ E/T)
```

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Simulated Annealing

- Theoretical guarantee:
  - If T decreased slowly enough, will converge to optimal state!
- Is this an interesting guarantee?
- Sounds like magic, but reality is reality:
  - The more downhill steps you need to escape, the less likely you are to every make them all in a row
  - People think hard about *ridge operators* which let you jump around the space in better ways

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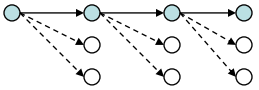
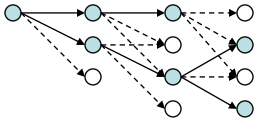
Beam Search

- Like greedy search, but keep K states at all times:



Greedy Search



Beam Search

- Variables: beam size, encourage diversity?
- The best choice in MANY practical settings
- Complete? Optimal?
- What criteria to order nodes by?

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